MULTIPLIERS

AND

THE HILBERT DISTRIBUTION

by

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0. INTRODUCTION. Let G be a locally compact Abelian topological group, let $\hat{G}$ be the dual group of G, let $L_p(G)$, $1 \leq p \leq \infty$, denote the usual L_p-spaces on G with respect to Haar measure, and denote by $A_p(G)$, $1 \leq p < \infty$, those f in $L_1(G)$ whose Fourier transform $\hat{f}$ belongs to $L_p(\hat{G})$. In this note we shall discuss some new and some known results concerning the multipliers for various pairs of the spaces $L_1(G)$, $L_p(G)$, and $A_p(G)$.

In particular, we shall prove that for any locally compact Abelian topological group G, the multipliers from $L_1(G)$ to $A_p(G)$ can be identified with the measures μ in $M(G)$, the Banach algebra of bounded, regular, complex-valued Borel measures on G, whose Fourier-Stieltjes transform $\hat{\mu}$ belongs to $L_p(\hat{G})$. Furthermore, denoting the group of complex numbers of absolute value one by Γ, we shall utilize the Hilbert distribution to give a new proof that for $1 < p < \infty$ the multipliers from $L_1(\Gamma)$ to $L_p(\Gamma)$ can be identified with $L_p(\Gamma)$, and prove that for $p > 2$ the Hilbert distribution does not determine a multiplier from $A_p(\Gamma)$ to $A_p(\Gamma)$. Among otherthings, the latter result implies for $p > 2$ the existence of f in $L_1(\Gamma)$ such that $\hat{f}$ belongs to $L_p(\mathbb{Z})$, $\mathbb{Z}$ being the group of integers, but the conjugate Fourier series of f is not the Fourier-Stieltjes series of any measure in $M(\Gamma)$.

Before we take up these results in detail we wish to set some notation and recall some definitions and basic results. A <u>Segal algebra</u> S on a locally compact Abelian topological group G is a translation invariant L_1-dense ideal in $L_1(G)$ that is a Banach algebra with respect to a norm $\|\cdot\|_S$ and such that for every g in S the mapping $s \to \tau_s g$ from G to S is continuous and $\|\tau_s g\|_S = \|g\|_S$. The symbol $\tau_s g$ denotes the translate of g by s, that is $\tau_s g(t) = g(t-s)$. It is easily verified for any locally compact

Abelian topological group G that $L_1(G)$ and $A_p(G)$, $1 \leq p < \infty$, are Segal algebras, and $L_p(G)$, $1 < p < \infty$, is a Segal algebra whenever G is compact. The Segal algebra norm in $A_p(G)$ is

$$\|f\|_{A_p} = \|f\|_1 + \|\hat{f}\|_p$$

$$= \int_G |f(t)|dt + \left(\int_{\hat{G}} |\hat{f}(\gamma)|^p d\gamma\right)^{1/p} .$$

If S_1 and S_2 are Segal algebras, then a <u>multiplier</u> from S_1 to S_2 is a continuous linear transformation T from S_1 to S_2 such that $T(f*g) = f * Tg = Tf * g$, $f,g \in S_1$, that is, T commutes with convolution. The space of multipliers from S_1 to S_2 is a Banach space under the operator norm and is denoted by $M(S_1,S_2)$. If $S_1 = L_1(G)$ or if $S_1 = S_2 = S$, then $M(S_1,S_2)$ is a commutative Banach algebra with identity. In the latter case we write $M(S_1,S_2) = M(S)$. Discussions of multipliers and Segal algebras can be found in [1,8,11,14], whereas the material from harmonic analysis used in the note is available in [4,5,7,16,18].

The symbol # is employed to indicate the completion of a proof.

1. MULTIPLIERS FROM $L_1(G)$ to $A_p(G)$. Given a locally compact Abelian topological group G and $1 \leq p < \infty$, we denote by $B_p(G)$ the space of all the measures μ in $M(G)$ such that $\hat{\mu}$ belongs to $L_p(\hat{G})$. It is easily seen that $B_p(G)$ is a commutative Banach algebra under convolution and with the norm

$$\|\mu\|_{B_p} = \|\mu\| + \|\hat{\mu}\|_p .$$

The symbol $\|\mu\|$ denotes the usual total variation norm of μ. Evidently, if G is discrete, then $B_p(G) = A_p(G) = L_1(G)$, and we will ignore this trivial situation. The multipliers from $L_1(G)$ to $A_p(G)$ are completely described by the following theorem:

THEOREM 1. *Let* G *be a nondiscrete locally compact Abelian topological group. If* $1 \leq p < \infty$ *and* $T : L_1(G) \to A_p(G)$, *then the following are equivalent:*

(i) $T \in M(L_1(G), A_p(G))$.

(ii) *There exists a unique* $\mu \in B_p(G)$ *such that* $Tf = \mu * f$, $f \in L_1(G)$.

Moreover, the mapping from $B_p(G)$ *to* $M(L_1(G), A_p(G))$ *defined by the formula* $Tf = \mu * f$, $f \in L_1(G)$, *is an isometric surjective algebra isomorphism.*

PROOF. Suppose $\mu \in B_p(G)$ and for every $f \in L_1(G)$ we have $Tf = \mu * f$. Clearly T is a linear transformation from $L_1(G)$ to $A_p(G)$ that commutes with convolution. Moreover, an elementary argument shows that T is continuous. Indeed, for every $f \in L_1(G)$ we have

$$\begin{aligned}\|Tf\|_{A_p} &= \|\mu * f\|_1 + \|\hat{\mu}\hat{f}\|_p \\ &\leq (\|\mu\| + \|\hat{\mu}\|_p)\|f\|_1 ,\end{aligned}$$

and so $\|T\| \leq \|\mu\|_{B_p}$. Considering T applied to an approximate identity for $L_1(G)$ of L_1-norm one, it is easily deduced that $\|T\| = \|\mu\|_{B_p}$.

Conversely, suppose $T \in M(L_1(G), A_p(G))$. Since $\|Tf\|_1 \leq \|Tf\|_{A_p}$, $f \in L_1(G)$, we deduce immediately that T determines a unique element of $M(L_1(G))$. Consequently, there exists a unique $\mu \in M(G)$ such that $Tf = \mu * f$, $f \in L_1(G)$, as the multipliers in $M(L_1(G))$ are all of this form [11,pp.2 and 3]. Now let $\{u_\alpha\} \subset L_1(G)$ be an approximate identity for $L_1(G)$ such that $\|u_\alpha\|_1 = 1$. If $1 < p < \infty$, then the inequalities

$$\|\hat{\mu}\hat{u}_\alpha\|_p \leq \|Tu_\alpha\|_{A_p} \leq \|T\|\|u_\alpha\|_1 = \|T\| ,$$

the reflexivity of $L_p(\hat{G})$, and the Banach-Alaoglu Theorem [12, pp. 254 and 258] imply that there exists a subnet $\{\hat{\mu}\hat{u}_\beta\}$ of $\{\hat{\mu}\hat{u}_\alpha\}$ and some $h \in L_p(\hat{G})$ such that $\{\hat{\mu}\hat{u}_\beta\}$ converges weakly to h. Since $\{\hat{\mu}\hat{u}_\beta\}$ converges uniformly to $\hat{\mu}$ on compact subsets of $\hat{G}$, it follows easily that $\hat{\mu} = h$ almost everywhere. Thus $\hat{\mu} \in L_p(\hat{G})$. A similar argument utilizing the weak-star compactness of the unit ball in $M(\hat{G})$ establishes the same conclusion when $p = 1$.

Therefore $\mu \in B_p(G)$, and the theorem is proved. #

Theorem 1 provides us with a simpler proof of a known description of $M(L_1(G), A_p(G))$, $1 \leq p \leq 2$, [10,p.660,13,p.41], which we state as Corollary 1. The proof is immediate an observing that if $1 \leq p \leq 2$, then the $\mu \in M(G)$ such that $\hat{\mu} \in L_p(\hat{G})$ are all absolutely continuous with respect to Haar measure on G [3,p.258, 13, p.38].

COROLLARY 1. <u>Let</u> G <u>be a nondiscrete locally compact Abelian topological group. If</u> $1 \leq p \leq 2$ <u>and</u> $T : L_1(G) \to A_p(G)$, <u>then the following are equivalent</u>:

(i) $T \in M(L_1(G), A_p(G))$.

(ii) <u>There exists a unique</u> $g \in A_p(G)$ <u>such that</u> $Tf = g * f$, $f \in L_1(G)$.

Moreover, the mapping from $A_p(G)$ *to* $M(L_1(G),A_p(G))$ *defined by the formula* $Tf = g * f$, $f \in L_1(G)$, *is an isometric surjective algebra isomorphism.*

In general, Corollary 1 fails for $p > 2$, that is, $A_p(G)$ is a proper subset of $B_p(G)$. For example, suppose $G = \Gamma$ and $2 < p < \infty$. Then for any ϵ, $0 < \epsilon < (p-2)/2p$, there exists a positive, continuous measure $\mu \in M(\Gamma)$ that is singular with respect to Haar measure on Γ and for which $\hat{\mu}(k) = O(k^{-\frac{1}{2}+\epsilon})$. It is readily verified that $\mu \in B_p(\Gamma)$, but $\mu \notin L_1(\Gamma)$. For a discussion of the existence of such a measure μ the reader is referred to [6,p.399].

Actually the results in [6] entail that $A_p(G)$ is a proper subset of $B_p(G)$ whenever G is a nondiscrete locally compact Abelian topological group and $p > 2$. This fact is not expressly stated in [6], although it has been observed previously without proof in [3,p.257]. The validity of the assertion when G is an infinite compact group follows immediately from the proof of Theorem 5.3 in [6], whereas for a noncompact nondiscrete group and $p \geq 4$ it is a simple corollary of Theorem 5.6 in [6]. A proof of the assertion when G is a noncompact nondiscrete group and $2 < p < 4$ is somewhat more complicated than for the preceding two cases, but still depends on the arguments in [6]. A complete discussion can be found in [9].

We shall have further comments on the existence of singular measures whose Fourier-Stieltjes transforms belong to $L_p(\hat{G})$, $2 < p < \infty$, following Corollary 2.

Prior to stating Corollary 2, we wish to make a preliminary observation. Obviously every Segal algebra is a module over $L_1(G)$

with respect to convolution. If S_1 and S_2 are Segal algebras, then by $\mathrm{Hom}_{L_1}(S_1,S_2)$ we mean the space of all continuous linear transformations T from S_1 to S_2 such that $T(f*g) = f * Tg$, $f \in L_1(G)$ and $g \in S_1$. The density of S_1 in $L_1(G)$ reveals that $M(S_1,S_2) = \mathrm{Hom}_{L_1}(S_1,S_2)$. This observation is made so that without further ado we may apply the results of [15] in the proof of Corollary 2. In particular, if X is a Banach space that is a module over $L_1(G)$, that is, X is an L_1-module, then $L_1(G)\otimes_{L_1} X$ will denote the L_1-module tensor product of $L_1(G)$ and X as defined in [15]. We say that $A_p(G)$ is the <u>dual space of an</u> L_1-<u>module</u> X provided $A_p(G)$ is isometically isomorphic to X^*, the dual space of X, and, on identifying $A_p(G)$ with X^* and denoting the module composition between $L_1(G)$ and X by $\circ$, we have $(f*g)(x) = g(f\circ x)$ for every $f \in L_1(G)$, $g \in A_p(G)$, $x \in X$.

COROLLARY 2. *If* G *is a nondiscrete locally compact Abelian topological group and* $p > 2$, *then*

(i) $A_p(G)$ *is not a prime ideal in* $M(G)$.

(ii) $A_p(G)$ *is not the dual space of an* L_1-*module*.

PROOF. Clearly $A_p(G)$ is an ideal in $M(G)$. From the remarks following Corollary 1 we know that there exists a singular measure $\mu \in M(G)$ such that $\hat{\mu} \in L_p(\hat{G})$. Thus, by Theorem 1, $\mu * f \in A_p(G)$ for every $f \in L_1(G)$, from which it follows that $A_p(G)$ is not a prime ideal since $A_p(G) \subsetneq L_1(G)$ [17,p.80].

In order to establish part (ii) of the corollary we suppose that $A_p(G)$ is the dual space of an L_1-module X. It is easily verified that $A_p(G)$ is an essential L_1-module [15,p.453], whence X is also an essential L_1-module [15,p.473]. Thus from [15,pp.461 and 462] we have

$$\begin{aligned} M(L_1(G),A_p(G)) &= \mathrm{Hom}_{L_1}(L_1(G),A_p(G)) \\ &\cong \mathrm{Hom}_{L_1}(L_1(G),X^*) \\ &\cong (L_1(G)\otimes_{L_1}X)^* \\ &\cong X^* \\ &\cong A_p(G)\,, \end{aligned}$$

contrary to the remarks following Corollary 1. Hence $A_p(G)$ is not the dual space of an L_1-module. #

If G is a nondiscrete locally compact Abelian topological group and $1 \leq p \leq 2$, then $A_p(G)$ is the dual space of an L_1-module as shown in [10,p.661, 13,pp.35,39-41]. This fact is a key step in the proof of Corollary 1 given in [10,13]. Actually, the same arguments reveal that $B_p(G)$ is the dual space of an L_1-module for all p, $1 \leq p < \infty$. Consequently, the proof of part (ii) of Corollary 2 shows that the existence of singular measures $\mu \in M(G)$ such that $\hat{\mu} \in L_p(\hat{G})$ is equivlent to $A_p(G)$ not being the dual space of an L_1-module.

Finally, we remark that a slight modification of the argument in [10,p.661] combined with the development in [13], especially Corollary 1.6 and Lemma 2.8, and Theorem 1 reveals that $\mathrm{Hom}_{L_1}(L_1(G),A_p(G)) \cong \mathrm{Hom}_{L_1}(L_1(G),B_p(G)) \cong B_p(G)$, $1 \leq p < \infty$.

2. MULTIPLIERS AND THE HILBERT DISTRIBUTION. On the circle group Γ the <u>Hilbert distribution</u> H is defined, as a distribution, by the formula

$$H = \sum_{n=-\infty}^{\infty} (-i \operatorname{sgn} n) e^{int} ,$$

where, as usual, $\operatorname{sgn} n = n/|n|$ if $n \neq 0$, and $\operatorname{sgn} n = 0$ if $n = 0$. If $f \in L_1(\Gamma)$, then $T_H f = H * f$ is defined as a distribution, and if the Fourier series of f is written formally as

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int} ,$$

then the distributional Fourier series of $T_H f$ is

$$\sum_{n=-\infty}^{\infty} (-i \operatorname{sgn} n) \hat{f}(n) e^{int} .$$

This series is generally called the <u>conjugate Fourier series</u> to the Fourier series of f. If the Fourier series of f is the cosine series

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nt ,$$

then the conjugate Fourier series is the sine series

$$\sum_{n=1}^{\infty} a_n \sin nt .$$

Evidently the Fourier transform of the distribution H is given by $\hat{H}(n) = -i \operatorname{sgn} n$. A famous theorem of M. Riesz asserts that if $1 < p < \infty$, then the formula $T_H f = H * f$, $f \in L_p(\Gamma)$, defines a continuous linear transformation from $L_p(\Gamma)$ to $L_p(\Gamma)$. Since T_H obviously commutes with convolution we see that $T_H \in M(L_p(\Gamma))$, $1 < p < \infty$. On the other hand, $T_H \notin M(L_1(\Gamma))$. Indeed, the cosine series

$$\sum_{n=2}^{\infty} \frac{\cos nt}{\ln n}$$

defines an element $f \in L_1(\Gamma)$, but

$$\sum_{n=2}^{\infty} \frac{\sin nt}{\ln n} ,$$

the Fourier series of $T_H f$, is not the Fourier-Stieltjes series of any measure on Γ, that is, $T_H f \notin M(\Gamma)$. A discussion of these results concerning the Hilbert distribution can be found in [4,pp.85-98].

In Theorem 3 we shall use the Hilbert distribution to give a new proof that $M(L_1(\Gamma),L_p(\Gamma))$ can be identified with $L_p(\Gamma)$, $1 < p < \infty$. A similar argument, which we shall sketch, yields yet another proof that $M(L_1(\Gamma),A_p(\Gamma))$ can be identified with $A_p(\Gamma)$, $1 \leq p \leq 2$. First, however, we need one preliminary result concerning the L_1-relative completion of a Segal algebra.

If G is a locally compact Abelian topological group and S is a Segal algebra on G, then the L_1-<u>relative completion</u> of S, denoted by $\tilde{S}$, is the space of those $f \in L_1(G)$ for which there exists a sequence $\{f_n\} \subset S$ such that $\sup_n\|f_n\|_S < \infty$ and $\lim_n\|f_n - f\|_1 = 0$. If $G = \Gamma$ and $S = L_p(\Gamma)$, $1 < p < \infty$, it is easily verified that $L_p(\Gamma)^{\sim} = L_p(\Gamma)$. The subject of relative completions of Segal algebras has been studied in [2]. We need only one general result from the theory which, loosely speaking, says that if S is a Segal algebra and $M(L_1(G),S) \subset L_1(G)$, then $M(L_1(G),S) \subset \tilde{S}$. More precisely we have the next theorem.

THEOREM 2. *Let G be a locally compact Abelian topological group, let S be a Segal algebra, and let $T \in M(L_1(G),S)$. If there exists some $g \in L_1(G)$ such that $Tf = g * f$, $f \in L_1(G)$, then $g \in \tilde{S}$.*

PROOF. Let $\{f_n\} \subset S$ be a sequence such that $\|f_n\|_1 = 1$ and $\lim_n\|g*f_n - g\|_1 = 0$ [12,pp.34-38]. Then $\{g*f_n\} \subset S$ as S is an ideal in $L_1(G)$ and

$$\begin{aligned}\sup_n \|g*f_n\|_S &= \sup_n \|Tf_n\|_S \\ &\leq \sup_n \|T\| \|f_n\|_1 \\ &= \|T\| ,\end{aligned}$$

whence $g \in \tilde{S}$. #

THEOREM 3. _If_ $1 < p < \infty$ _and_ $T : L_1(\Gamma) \to L_p(\Gamma)$, _then the following are equivalent:_

(i) $T \in M(L_1(\Gamma), L_p(\Gamma))$.

(ii) _There exists a unique_ $g \in L_p(\Gamma)$ _such that_ $Tf = g * f$, $f \in L_p(\Gamma)$.

Moreover, the mapping from $L_p(\Gamma)$ _to_ $M(L_1(\Gamma), L_p(\Gamma))$ _defined by the formula_ $Tf = g * f$, $f \in L_1(\Gamma)$, _is an isometric surjective algebra isomorphism._

PROOF. Obviously part (ii) implies part (i). Suppose $T \in M(L_1(\Gamma), L_p(\Gamma))$. Since $\|Tf\|_1 \leq \|Tf\|_p$, $f \in L_1(\Gamma)$, we see that $T \in M(L_1(\Gamma))$. Consequently, there exists a unique $\mu \in M(\Gamma)$ such that $Tf = \mu * f$, $f \in L_1(\Gamma)$ [11,pp.2 and 3]. Moreover, since $T_H \in M(L_p(\Gamma))$, we see at once that $T_H \circ T \in M(L_1(\Gamma), L_p(\Gamma))$, and so repeating the argument just used we deduce that $H * \mu \in M(\Gamma)$. A form of the F. and M. Riesz Theorem [4,p.94,18,I,p.285] then implies that μ is absolutely continuous. Hence there exists a unique $g \in L_1(\Gamma)$ such that $Tf = g * f$, $f \in L_1(\Gamma)$, whence, by Theorem 2, $g \in L_p(\Gamma)^{\sim} = L_p(\Gamma)$. Thus part (i) implies part (ii).

The remainder of the proof is standard. #

Note that the proof of Theorem 3 fails for $p = \infty$ as $L_\infty(\Gamma)$

is not a Segal algebra and $T_H \notin M(L_\infty(\Gamma))$ [4,p.91]. Proofs of Theorem 3 that are valid for all p, $1 < p \leq \infty$, are available in [4,p.255, 11,pp.67-69].

The proof of Theorem 3 can be easily modified to give yet another proof of Corollary 1 when $G = \Gamma$. All that is needed is to observe that $T_H \in M(A_p(\Gamma))$, $1 \leq p \leq 2$, and that $A_p(\Gamma)^\sim = A_p(\Gamma)$, $1 \leq p < \infty$. The first fact is immediate since every bounded function on $\hat{\Gamma} = \mathbb{Z}$ determines a multiplier of $A_p(\Gamma)$, $1 \leq p \leq 2$, [11,p.207], and the second fact is the content of the next theorem.

THEOREM 4. _If_ $1 \leq p < \infty$, _then_ $A_p(\Gamma)^\sim = A_p(\Gamma)$.

PROOF. Suppose $1 < p < \infty$ and $f \in A_p(\Gamma)^\sim$. Then there exists a sequence $\{f_n\} \subset A_p(\Gamma)$ such that $\sup_n \|f_n\|_{A_p} < \infty$ and $\lim_n \|f_n - f\|_1 = 0$. Since $\|\hat{f}_n\|_p \leq \|f_n\|_{A_p}$, the reflexivity of $L_p(\mathbb{Z})$ and Helly's Selection Theorem [12,p.261] imply that there exists a subsequence $\{\hat{f}_{n_k}\}$ of $\{\hat{f}_n\}$ and an $h \in L_p(\mathbb{Z})$ such that $\{\hat{f}_{n_k}\}$ converges weakly to h. But $\{\hat{f}_{n_k}\}$ converges uniformly to $\hat{f}$, whence $h = \hat{f} \in L_p(\mathbb{Z})$. Therefore $f \in A_p(\Gamma)$ and $A_p(\Gamma)^\sim = A_p(\Gamma)$.

When $p = 1$ a similar argument and the fact that $L_1(\mathbb{Z}) = M(\mathbb{Z})$ shows that $A_1(\Gamma)^\sim = A_1(\Gamma)$. #

Preceding Theorem 4 we observed that $T_H \in M(A_p(\Gamma))$, $1 \leq p \leq 2$. This is not the case if $p > 2$.

THEOREM 5. _If_ $p > 2$, _then_ $T_H \notin M(A_p(\Gamma))$.

PROOF. Suppose $T_H \in M(A_p(\Gamma))$. If $\mu \in B_p(\Gamma)$, then, by Theorem 1, the formula $Tf = \mu * f$, $f \in L_1(\Gamma)$, defines an element

T of $M(L_1(\Gamma), A_p(\Gamma))$. But then the assumption that $T_H \in M(A_p(\Gamma))$ obviously implies that $T_H \circ T \in M(L_1(\Gamma), A_p(\Gamma))$, whence, appealing once more to Theorem 1, we conclude that $H * \mu \in B_p(\Gamma) \subset M(\Gamma)$. Consequently, we deduce from the F. and M. Riesz Theorem [4,p.94, 18,I,p.285] and Theorem 4 that $\mu \in A_p(\Gamma)$. Hence $A_p(\Gamma) = B_p(\Gamma)$, contrary to the first remark following Corollary 1.

Therefore $T_H \notin M(A_p(\Gamma))$. #

We noted at the beginning of this section that there exist $f \in L_1(\Gamma)$ such that $T_H f \notin M(\Gamma)$. The example given of such an f, namely the cosine series

$$\sum_{n=2}^{\infty} \frac{\cos nt}{\ln n},$$

evidently does not belong to $A_p(\Gamma)$, $1 \leq p < \infty$. The proof of Theorem 5 immediately shows that if $p > 2$, then there exists some $f \in A_p(\Gamma)$ such that $T_H f \notin M(\Gamma)$.

Theorem 5 has a number of other simple consequences which we collect in the next corollary. It seems likely that most of these results are known, but we include them as examples of the utility of Theroem 5 and its method of proof.

COROLLARY 3. _If_ $p > 2$, _then_:

(i) _There exists some_ $f \in L_1(\Gamma)$ _such that_ $\hat{f} \in L_p(\mathbb{Z})$ _and the conjugate Fourier series of_ f _is not the Fourier-Stieltjes series of any measure in_ $M(\Gamma)$.

(ii) _There exists some_ $f \in L_1(\Gamma)$ _such that_ $\hat{f} \in L_p(\mathbb{Z})$ _and_ $\Sigma_{n=0}^{\infty} \hat{f}(n) e^{int}$ _is not the Fourier-Stieltjes series of any measure in_ $M(\Gamma)$.

(iii) <u>There exists some</u> $f \in L_1(\Gamma)$ <u>such that</u> $\hat{f} \in L_p(\mathbb{Z})$ <u>and the Fourier series of</u> f <u>does not converge in</u> L_1-<u>norm to</u> f.

(iv) <u>There exists some</u> $f \in L_1(\Gamma)$ <u>such that</u> $\hat{f} \in L_p(\mathbb{Z})$ <u>and</u> $|f|\log^+|f| \notin L_1(\Gamma)$.

PROOF. The proof of part (i) is contained in the remarks preceding the corollary, whereas part (ii) follows at once from part (i) on considering the mapping defined by convolution with the distribution $(1+\delta+iH)/2$. The symbol δ denotes the measure on Γ with unit mass concentrated at $z = 1$.

Since $T_H \notin M(A_p(\Gamma))$ for $2 < p < \infty$, it follows that there exists some $f \in A_p(\Gamma)$ such that the Fourier series of f does not converge to f in the norm of $A_p(\Gamma)$. This assertion is an immediate consequence of a theorem of Katznelson [7,p.49]. Moreover, sinc $\hat{f} \in L_p(\mathbb{Z})$, we see that

$$\lim_N \|\hat{f} - (\sum_{|n|\le N} \hat{f}(n)e^{int})^\wedge\|_p = \lim(\sum_{|n|>N} |\hat{f}(n)|^p)^{1/p} = 0,$$

whence we conclude that the Fourier series of f does not converge to f in L_1-norm. This proves part (iii), and part (iv) follows on observing that if $|f|\log^+|f| \in L_1(\Gamma)$, then the Fourier series of f converges in L_1-norm to f [18,I,p.267]. #

The existence of $f \in L_1(\Gamma)$ that satisfies the conclusions of parts (i) - (iv) of Corollary 3 is, of course, well known [4,pp.90 and 102, 7,pp.24, 64, and 66, 18,I,pp.184, 253, 254, and 267]. Corollary 3 may be of interest because given $p > 2$ it establishes the existence of such f for which $\hat{f} \in L_p(\mathbb{Z})$.

Given $1 < p < 2$, if $f \in L_p(\Gamma)$, then the conjugate Fourier series of f is the Fourier series of a function in $L_p(\Gamma)$, and the Fourier series of f converges to f in the L_1-norm. These observations are just the M. Riesz Theorem and one of its consequences [4,pp.94 and 100]. Since $L_p(\Gamma) \subset A_{p'}(\Gamma)$, $1/p + 1/p' = 1$, these remarks together with Corollary 3 immediately show that $L_p(\Gamma)$ is a proper subset of $A_{p'}(\Gamma)$. Utilizing this observation we can give a simple proof that $L_p(\Gamma)^\wedge$ is a proper subset of $L_{p'}(\mathbb{Z})$. Other proofs can be found, for example, in [4,p.227, 7,p.101, 18,II, p.102].

COROLLARY 4. _If_ $1 < p < 2$ _and_ $1/p + 1/p' = 1$, _then_ $L_p(\Gamma)^\wedge$ _is a proper subset of_ $L_{p'}(\mathbb{Z})$.

PROOF. It follows at once from the Hausdorff-Young Theorem [4,p.145] and the preceding remarks that $L_p(\Gamma)^\wedge \subsetneqq A_{p'}(\Gamma)^\wedge \subset L_{p'}(\mathbb{Z})$. #

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